Homework #8

[Your name]

[Total - 100 pts]

Question 1: Let $X_1, \ldots, X_{25} \sim N(\mu, 5^2)$ where $n = 25$. Test the following hypotheses.

$$H_0 : \mu = 60 \quad \text{vs} \quad H_1 : \mu > 60.$$

(a) Let Φ denote the CDF for a standard normal distribution, $N(0, 1)$. Write the significance probability for the rejection region $\bar{X} > c$ (c is some constant) in terms of ϕ.

(b) Determine the value of c for which the rejection region $\bar{X} > c$ results in a test with a size of $\alpha = 0.05$ (need R/Python/software to compute).

(c) Write the general power function, $\gamma_c(\mu)$, for the rejection region $\bar{X} > c$. Then, calculate the power of the test $\gamma_c(65)$ using the value of c from part (b). [*Need R/Python/software to compute*]

[1.a] Solution ■

[1.b] Solution ■

[1.c] Solution ■

Question 2: We define the significance probability as the probability of observing a result that contradicts the null hypothesis H_0 more than the given test statistic. This implies the smallest significance level at which the observed result can reject the null hypothesis H_0. Let $X_1, \ldots, X_7 \sim$ Bernoulli(p). Suppose we want to test the following hypotheses,

$$H_0 : p = 0.5 \quad \text{vs} \quad H_1 : p > 0.5,$$

and we observe $x_1 + \cdots + x_7 = 6$. Calculate the significance probability. [*Hint:* $Y = X_1 + \ldots + X_7$ *follows a well known distribution that can be used to construct a rejection region.*]

[2] Solution ■

Question 3: Let $X_1, \ldots, X_n \sim \text{Exp}(\theta)$, where $0 < \theta < \infty$, suppose we want to test the following hypothesis by using the likelihood ratio test

$$H_0 : \theta \leq \theta_0 \quad \text{vs} \quad H_1 : \theta > \theta_0 \quad (\theta_0 \text{ is a given value})$$

at a significance level $\alpha (0 < \alpha < 1)$. Find λ^* as defined in Definition 8.6(a). The density of $\text{Exp}(\theta)$ is

$$f_X(x \mid \theta) = \frac{1}{\theta} \exp\Big(-\frac{x}{\theta}\Big), \quad x \in [0, \infty).$$

[*Hint: the answer is a piece-wise function depending on how* θ_0 *relates to* $\bar{X}$]

[3] Solution ■

Question 4: Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ with $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$ (where $n \geq 2$), we want to test the following hypotheses at the significance level α (where $0 < \alpha < 1$).

$$H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_1 : \mu \neq \mu_0 \quad (\mu_0 \text{ is a given value})$$

(a) Recall that the maximum likelihood estimates for μ and σ^2 are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

Find the likelihood ratio test statistic λ^*.

(b) Show that the rejection region for the likelihood ratio test can be written as

$$\frac{|\bar{X} - \mu_0|}{S/\sqrt{n}} \geq c,$$

where $S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}$.

(c) What value of c leads to a significance level of α? [*Hint: under* H_0, $\frac{\bar{X} - \mu_0}{S/\sqrt{n}}$ *follows a well-known distribution, so this should not require much calculation and can be written in terms of a quantile of this well-known distribution*]

[4.a] Solution ■

[4.b] Solution ■

[4.c] Solution ■