Homework #6

[Your name]

`[Total - 100 pts]`

Question 1: For each of the following, (i) verify each distribution belongs to the exponential family, (ii) find the MLE for θ, and (iii) find the limiting distribution of the MLE.

(a) Beta distribution: $X_1, \ldots, X_n \overset{iid}{\sim} \text{Beta}(\theta, 1)$, $\theta > 0$. [*Hint: note that* $-\theta \log(X) \sim \mathit{Exp}(1)$ *if* $X \sim \mathit{Beta}(\theta, 1)$]

(b) Pareto distribution: $X_1, \ldots, X_n \overset{iid}{\sim} \text{Pareto}(1, \theta)$, $\theta > 2$. [*Hint: note that* $\theta \log(X) \sim \mathit{Exp}(1)$ *if* $X \sim \mathit{Pareto}(1, \theta)$]

[1.a] Solution ■

[1.b] Solution ■

Question 2: Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Uniform}(-\theta, \theta)$ where $\theta > 0$. Answer the following:

(a) Show that the statistic $Y = \max\{|X_i|\}$ for $1 \le i \le n$ and $\theta \in (0, +\infty)$ is a sufficient statistic.

(b) Denote the order statistics as $X_{(1)} < \cdots < X_{(n)}$ for $n \ge 2$, and define an estimator for θ as:

$$\hat{\theta} = c_n \cdot \left(X_{(n)} - X_{(1)}\right)$$

Find the value of c_n such that the estimator is unbiased for θ (i.e., $\mathbb{E}(\hat{\theta}) = \theta$).

[2.a] Solution ■

[2.b] Solution ■

Question 3: Find a sufficient statistic:

(a) $X_1, \ldots, X_n \overset{iid}{\sim} \text{Exp}(\theta)$, for $\theta > 0$, where the pdf is given by

$$\text{pdf}(x; \theta) = \frac{1}{\theta} e^{-x/\theta} \cdot \mathbf{1}\{x \in (0, \infty)\}.$$

(b) $X_1, \ldots, X_n \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$, for $\alpha > 0$ and $\beta > 0$, where the pdf is given by

$$\text{pdf}(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \cdot \mathbf{1}\{x \in (0, \infty)\}.$$

[3.a] Solution ■

[3.b] Solution ■

Question 4: Consider an exponential distribution $\text{Exp}(\mu, \sigma)$ with two parameters, where $-\infty < \mu < +\infty$ and $\sigma > 0$. The marginal pdf is given by

$$\text{pdf}(x; \mu, \sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma} \cdot \mathbf{1}\{x \in (\mu, \infty)\}.$$

Suppose that you are unable to observe the individual random samples, but you can observe their order statistics $X_{(1)} < \cdots < X_{(r)}$ for $1 \leq r < n$.

(a) Does $\text{Exp}(\mu, \sigma)$ belong to the exponential family? Explain.

(b) Find a sufficient statistic for (μ, σ), where $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$.

[4.a] Solution ■

[4.b] Solution ■