Homework #3

[Your Name]

[Total - 100 pts]

(Useful information) The following identity will be necessary for this homework:

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a, \quad a \in \mathbb{R}$$

Question 1: Suppose a random sample $X_1, \ldots, X_n$ follows a distribution below with the probability density function

$$f(x;\theta) = e^{-(x-\theta)} \cdot \mathbf{1}\{x > \theta\},$$

where θ is a real number. Let the statistic $Y_n = \min_{1\le i\le n} X_i$. We want to show that $Y_n \xrightarrow{P} \theta$.

(a) Find the cumulative distribution function of X_1, $\text{cdf}_{X_1}(x) = P(X_1 \le x)$.

(b) Find the probability density function of Y_n.

(c) Using Theorem 3.3, show $Y_n \xrightarrow{P} \theta$.

[1.a] Solution ■

[1.b] Solution ■

[1.c] Solution ■

Question 2: Suppose $X_1, \ldots, X_n \overset{iid}{\sim} \text{Beta}(1, \alpha)$ for $\alpha > 0$. We ultimately want to find the asymptotic distribution of $n^{1/\alpha}(1 - Y_n)$, where $Y_n = \max_{1\le i\le n} X_i$.

(a) Find the probability density function of Y_n.

(b) Let $W_n = n^{1/\alpha}(1 - Y_n)$, find the probability density function of W_n by using the variable transformation.

(c) Find the cumulative distribution function of W_n. For each value of the cdf, where does it converge?

(d) Let W be the asymptotic distribution that has the cdf above. Find the probability density function of W.

[2.a] Solution ■

[2.b] Solution ■

[2.c] Solution ■

[2.d] Solution ■

Question 3: Given the probability density function $f(x) = \alpha x^{-\alpha-1} \cdot \mathbf{1}\{x > 1\}$ with $\alpha > 0$, find the probability density function of the asymptotic distribution of $n^{-1/\alpha} Y_n$, where $Y_n = \max_{1\leq i\leq n} X_i$. Follow the steps in *Question 2*.

[3] Solution ■

Question 4: Given $X_1, \ldots, X_n \overset{iid}{\sim} \text{Exp}(1)$, find the asymptotic distribution of $Y_n - \log n$, where $Y_n = \max_{1\leq i\leq n} X_i$. Again, follow the steps in *Question 2*.

[4] Solution ■

Question 5: Suppose a random sample $X_1, \ldots, X_n \overset{iid}{\sim} \text{Beta}(\alpha, 1)$ with $\alpha > 0$. We consider $Y_n = \min_{1\leq i\leq n} X_i$. Find the value of r such that $n^r Y_n$ has an asymptotic distribution.
Hint: for some fixed $a > 0$

$$\lim_{n\to\infty} \left(1 - \frac{a}{n^r}\right)^n = \begin{cases} 0, & r < 1 \\ e^{-a} & r = 1 \\ 1, & r > 1 \end{cases}$$

[5] Solution ■

Question 6: Suppose $U_1, \ldots, U_n \overset{iid}{\sim} \text{Unif}(0, 1)$. Define $X_n = \min_{1\leq i\leq n} U_i$ and $Y_n = \max_{1\leq i\leq n} U_i$. Set $R_n = Y_n - X_n$.

(a) Prove that $X_n \xrightarrow{P} 0$ and $Y_n \xrightarrow{P} 1$.

(b) Find the limiting distribution of $n(1 - R_n)$ (i.e., what does the CDF of $n(1 - R_n)$ converge to).

[6.a] Proof □

[6.b] Solution ■