Homework #2

Your Name

`[Total - 100 pts]`

Question 1: Suppose a random sample $X_1, X_2, \ldots, X_n$ follow a distribution with the probability density function:

$$f(x) = 3x^2 \cdot \mathbf{1}\{x \in (0,1)\}$$

(a) Find the cdf of $Y = \min_{1 \le i \le n} X_i$, and then use the derivative of the cdf to find pdf of Y. *[Hint: what is an equivalent probability to $P(Y \le y)$ in terms of i.i.d. $X_1, \ldots, X_n$?]*

(b) Find the cdf of $Y = \max_{1 \le i \le n} X_i$, and then use the derivative of the cdf to find pdf of Y. *[Hint: what is an equivalent probability to $P(Y \le y)$ in terms of i.i.d. $X_1, \ldots, X_n$?]*

[1.a] Solution ■

[1.b] Solution ■

Question 2: Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ (where $n \ge 2$) be order statistics based on a random sample $X_1, \ldots, X_n \sim \text{Exp}(1)$. We want to find the probability density function of $Y = X_{(n)} - X_{(1)}$.

(a) Find the joint probability density function of $X_{(1)}$ and $X_{(n)}$.

(b) Let $X = X_{(1)}$. Suppose that we define a function u such that

$$u : \begin{cases} X = X_{(1)} \\ Y = X_{(n)} - X_{(1)} \end{cases}$$

Find the inverse function of u, the Jacobian matrix $\frac{\partial u^{-1}(x_{(1)}, x_{(n)})}{\partial(x,y)}$, and its determinant.

(c) Find the marginal probability density function of Y.

[2.a] Solution ■

[2.b] Solution ■

[2.c] Solution ■

Question 3: Let $U_{(1)} < U_{(2)} < \cdots < U_{(n)}$ be the order statistics of a sample of size $n \geq 2$ from a uniform distribution $U(0, 1)$. Find the probability density function of $Y = U_{(n)} - U_{(1)}$. Follow the steps as that of *Question 2*.

[3] Solution ■

Question 4: Consider the order statistics $X_{(1)} < \cdots < X_{(n)}$ based on a random sample $X_1, \ldots, X_n \sim \text{Exp}(1)$. Define

$$Z_1 = nX_{(1)}, \quad Z_2 = (n-1)\left(X_{(2)} - X_{(1)}\right), \quad \ldots, \quad Z_n = X_{(n)} - X_{(n-1)}.$$

Now we want to show that $Z_1, Z_2, \ldots, Z_n$ are independent and each follows an exponential distribution $\text{Exp}(1)$.

(a) Let $Y = (Y_1, Y_2, \ldots, Y_n)^{\mathrm{T}} = (X_{(1)}, X_{(2)}, \ldots, X_{(n)})^{\mathrm{T}}$. Find the probability density function of Y.

(b) Let $Z = (nX_{(1)}, (n-1)(X_{(2)} - X_{(1)}), \ldots, X_{(n)} - X_{(n-1)})^{\mathrm{T}}$ and

$$\mathcal{Y} = \{y = (y_1, \ldots, y_n)^{\mathrm{T}} : 0 < y_1 < \cdots < y_n\},$$
$$\mathcal{Z} = \{z = (z_1, \ldots, z_n)^{\mathrm{T}} : z_1 > 0, \ldots, z_n > 0\}.$$

If we define a function $u : \mathcal{Y} \to \mathcal{Z}$ such that $Z = u(Y)$, show u is a bijection.

(c) Calculate the Jacobian determinant $\det\left(\frac{\partial u^{-1}(z_j)}{\partial z_i}\right)$ of the inverse function u^{-1}.

(d) Calculate the probability density function of Z and arrive at the initially proposed conclusion (i.e., $Z_1, Z_2, \ldots, Z_n$ are independent and each follows an exponential distribution $\text{Exp}(1)$).

[4.a] Solution ■

[4.b] Solution ■

[4.c] Solution ■

[4.d] Solution ■