Homework #1

Your Name

`[Total - 100 pts]`

Question 1: If X_1 and X_2 are independent random variables following Poisson(λ_1) and Poisson(λ_2) distributions respectively:

(a) Find the probability mass function for $Y = X_1 + X_2$. *[Hint: recall the binomial expansion* $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$*]*

(b) From (a), determine the distribution of Y.

(c) Recall the moment generating function of a random variable and its properties. Using the two properties discussed in class, determine (b) again.

[1.a] Solution ■

[1.b] Solution ■

[1.c] Solution ■

Question 2: The joint probability density function of the random variable $X = (X_1, X_2)^t$ is

$$\mathrm{pdf}_X(x_1, x_2) = \frac{1}{\pi} \mathbf{1}_{\{(x_1,x_2):0<x_1^2+x_2^2<1 \text{ and } x_1\neq 0, x_2\neq 0\}}.$$

We want to find the joint probability density function of $Y = (Y_1, Y_2)^t$ defined as

$$Y_1 = X_1^2 + X_2^2, \quad Y_2 = \frac{X_1^2}{X_1^2 + X_2^2}.$$

(a) Let the transformation $u : \mathbb{X} \to \mathbb{Y}$. Define the domain, $\mathbb{X}$, and the range, $\mathbb{Y}$, of u.

(b) Let u be an m-to-1 function. Write (X_1, X_2) as a function of (Y_1, Y_2) and determine m.

(c) Write out the m disjoint sets whose union equals $\mathbb{X}$ and where u is one-to-one from each of the m sets to $\mathbb{Y}$.

(d) Find the joint probability density function of $Y = (Y_1, Y_2)^t$. Show that Y_1, Y_2 are independent and $Y_1 \sim \mathrm{Unif}(0, 1)$ and $Y_2 \sim \mathrm{Beta}(1/2, 1/2)$ *[Useful info:* $\Gamma(1/2) = \sqrt{\pi}$*]*

[2.a] Solution ■

[2.b] Solution ■

[2.c] Solution ■

[2.d] Solution ■

Question 3: Let the joint probability density function of random variables X_1 and X_2 be

$$f(x_1, x_2) = 8x_1x_2\mathbf{1}_{(0<x_1<x_2<1)}.$$

(a) Find the joint probability density function of $Y_1 = X_1/X_2$ and $Y_2 = X_2$.

(b) Determine whether Y_1 and Y_2 are independent.

[3.a] Solution ■

[3.b] Solution ■

Question 4: Recall the definition of the χ^2 distribution. Let independent random variables $X_1, \ldots, X_r$ follow a standard normal distribution $\mathcal{N}(0, 1)$. We want to prove that $Y = X_1^2 + \cdots + X_r^2$ follows Gamma$(r/2, 2)$.

(a) Suppose X follows $\mathcal{N}(0, 1)$. Show that $Z = X^2$ follows Gamma$(1/2, 2)$.

(b) By using a result of Example 1.5, prove $Y \sim$ Gamma$(r/2, 2)$. This concludes that Gamma$(r/2, 2) = \chi^2(r)$.

[4.a] Proof □

[4.b] Proof □

Question 5: Suppose two random samples from normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ with $X_{11}, X_{12}, \ldots, X_{1n_1}$ and $X_{21}, X_{22}, \ldots, X_{2n_2}$ respectively, and assume that the two samples are independent. By using the definition of F-distribution, show that

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(n_1 - 1, n_2 - 1),$$

where $\bar{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i, \quad S_i^2 = \sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2/(n_i - 1) \quad (i = 1, 2)$. You may use theorems in the lecture notes without proving them as long as they are cited.

[5] Proof $\square$

Question 6: Suppose the random variable X follows an $F(r_1, r_2)$ distribution. Show that

$$Y = \frac{(r_1/r_2)X}{1 + (r_1/r_2)X}$$

follows a Beta$(r_1/2, r_2/2)$ distribution.

[6] Proof $\square$